4.

Unruh Radiation

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Read: Carroll GR Ch. 9 (QFT in curved spacetime)
QGBH §3 and §5

Reduced Density Matrix

Suppose a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Given a density matrix $\rho : \mathcal{H} \to \mathcal{H}$,

$$\rho_A = \mathrm{tr}_B\, \rho$$

This called the "reduced density matrix".

ex 2 qubits $\quad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

Pure state $\quad |\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \Big)$ (Bell pair)

$$\rho = |\psi\rangle\langle\psi|$$

$$\rho_1 = \mathrm{tr}_2\, |\psi\rangle\langle\psi|$$

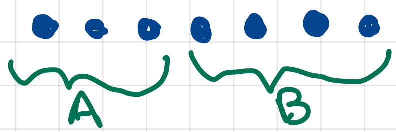
$$= \langle\uparrow_2|\rho|\uparrow_2\rangle + \langle\downarrow_2|\rho|\downarrow_2\rangle$$

$$= \frac{1}{2}\Big(|\uparrow\rangle + |\downarrow\rangle \Big)$$

maximally mixed

(50/50 with only classical uncertainty)

<u>ex. 2</u> Spin chain



$$\mathcal{H} = \bigotimes^N \mathcal{H}_{spin}$$

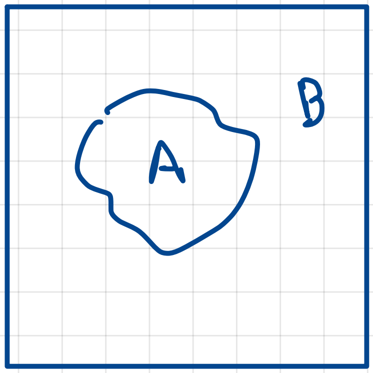
$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Thus in quantum mechanics we can speck of the density matrix for a spatially-defined "subregion" of a spin chain.

* For any operators $\mathcal{O}_i$ on $\mathcal{H}_A$,

$$\mathrm{Tr}\, \mathcal{O}_1 \mathcal{O}_2 \cdots \rho = \mathrm{Tr}\, \mathcal{O}_1 \mathcal{O}_2 \cdots \rho_A$$

In QFT, consider

Space = 

It is almost true that

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Caveats:

① UV divergences

We must put a small but finite separation between A, B for them to factorize.

② gauge theory

Gauge theories have link variables and Gauss constraints.

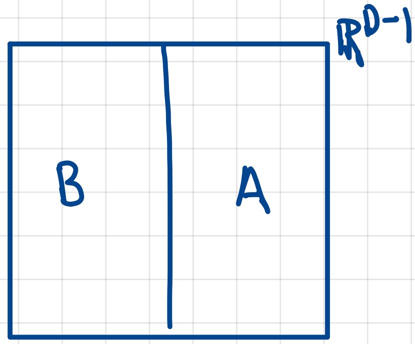
There are techniques to deal with both of these that do not change our conclusions, so for now we can ignore both subtleties.

Rindler Space

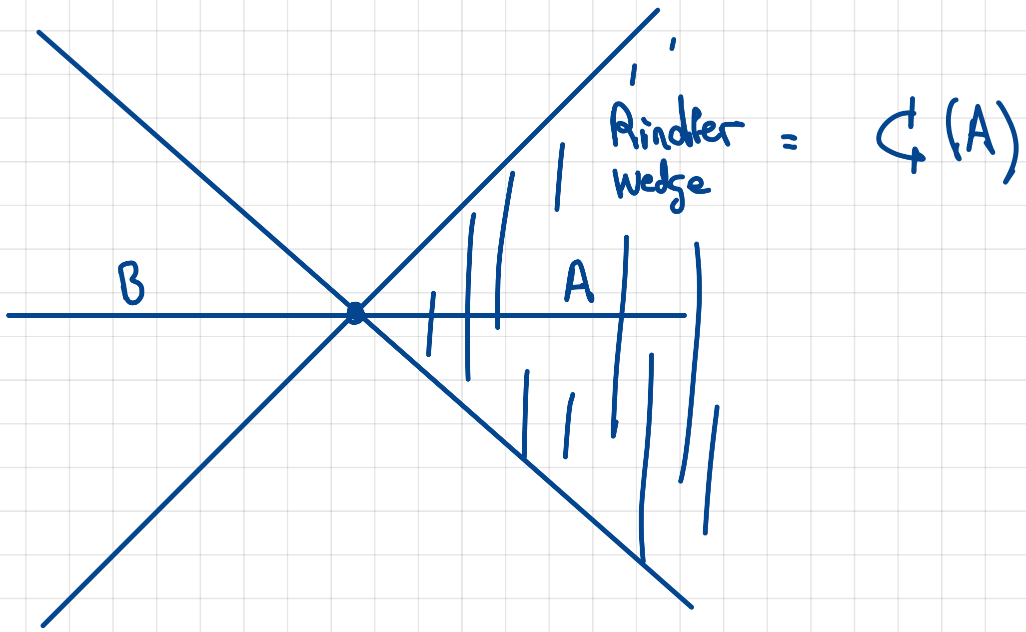
QFT in $\mathbb{R}^{D-1,1}$

Divide

Space =

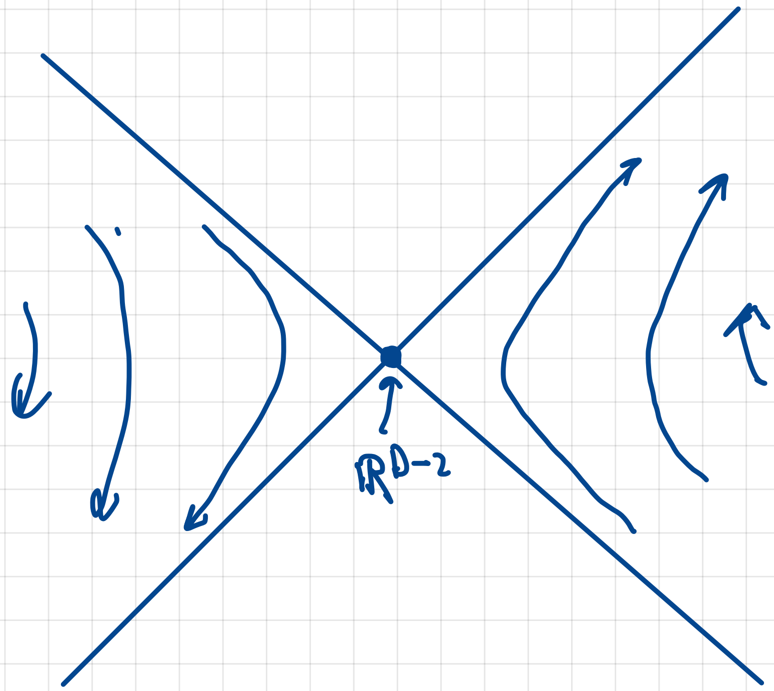


In Lorentzian:



Q: What is ρ_A?

First, the claim:



$\mathbb{R}^{D-1,1}$

Boost

$$\xi_{(y)} = t\partial_y + y\partial_t$$
$$= t\hat{y} + y\hat{t}$$

Let $K \equiv$ generator of $\hat{y}$ boosts (boost charge)

Claim:

$$\rho_A = e^{-2\pi K_A}$$

$\boxed{\text{Thermal}}$ (w.r.t. boosts) @ $T = \frac{1}{2\pi}$

Let's write an explicit formula for K:

Stress tensor $T_{\mu\nu}$

Hamiltonian $H = \int_{\text{space}} T^{00}$

$$= \int_{\text{space}} d^{D-1}x \sqrt{h}\, u^\mu u^\nu T_{\mu\nu}$$

timelike normal $u = \partial_t$

For a Killing vector ξ, charge

$$Q[\xi] = \int d\Sigma^\mu T_{\mu\nu} \xi^\nu$$

In particular,

$$K \equiv Q[\xi_y]$$

Σ @ $t=0$
$$= \int_{-\infty}^{\infty} dy \int d^{D-2}\vec{x}_\perp \, y \, T_{tt}$$

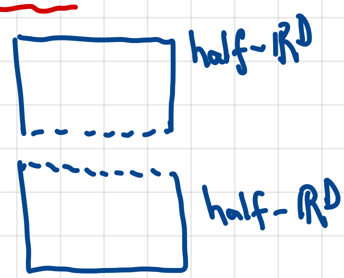
$$K_A = \int_0^{\infty} dy \int d^{D-2}\vec{x}_\perp \, y \, T_{tt}$$

This is the "energy" that defines our thermal state.

Note we could choose any Σ_A to write the K_A integral.

Euclidean Derivation

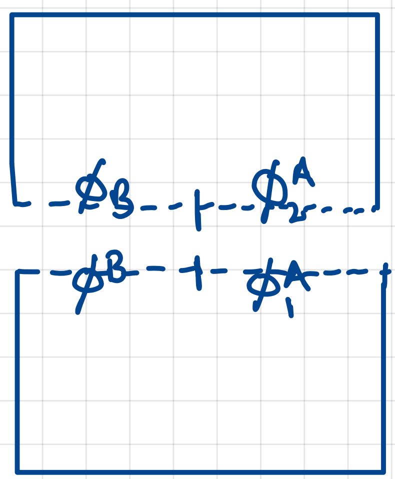
$$\rho = |0\rangle\langle 0| =$$



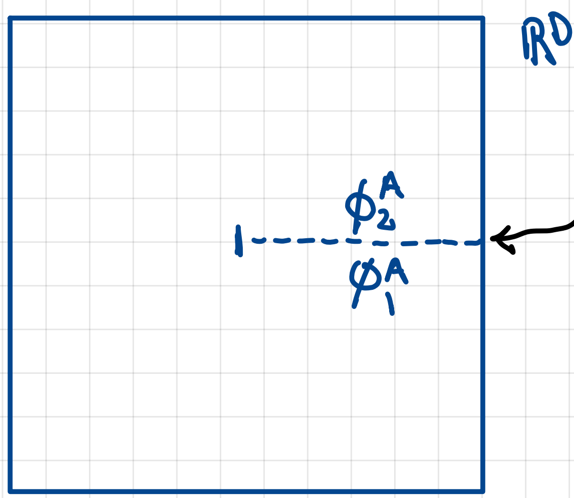
$$\rho_A = \mathrm{tr}_B\, \rho$$

$$\langle \phi_2^A | \rho_A | \phi_1^A \rangle = \sum_{\phi^B} \langle \phi_2^A, \phi^B | \rho | \phi_1^A, \phi^B \rangle$$

$$= \sum_{\phi_B}$$



$$=$$

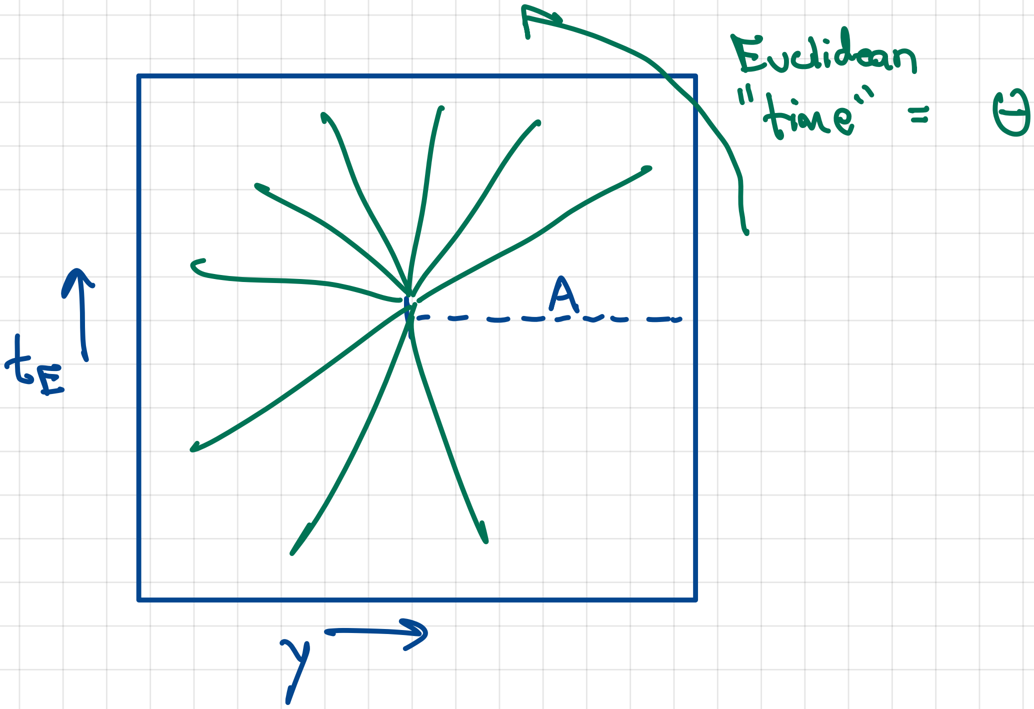


Different boundary conditions on top and bottom of cut.

Again, trace "glues" the manifold.

Now the key step.

Re-slice this path integral radially:



$$= \langle \phi_2^A | \, e^{-2\pi Q[\partial_\theta]} \, | \phi_1^A \rangle$$

$$Q[\partial_\theta] = \int_A d\Sigma^\mu T_{\mu\nu} \hat{\theta}^\nu$$

$\Rightarrow$

$$\rho_A = e^{-2\pi Q[\partial_\theta]}$$

This is almost what we wanted, but Euclidean. Let's check it is same as Lorentzian boost generator.

The Euclidean rotation is

$$\partial_\theta = t_E \partial_y - y \partial_{t_E}$$

Wick rotate $t_E = it$

$$\rightarrow \quad i\left(t\partial_y + y\partial_t\right)$$

The normal also rotates $u = \partial_{t_E} \rightarrow -i\partial_t$

$\therefore$

$$Q[\partial_\theta] = K_A$$

$$\rho_A = e^{-2\pi K_A}$$

Note boost ∂_θ, unlike translation ∂_t is a dimensionless coordinate. So K_A is dimensionless; so Temperature is dimensionless.

(T is dimensionless)

Lorentzian Interpretation

Euclidean coordinates:

$$ds^2 = dR^2 + R^2 d\theta^2 + d\vec{x}_\perp^{\,2} \qquad \text{(cylindrical)}$$

Lorentzian:

$$\theta \to i\eta$$

$$ds^2 = -R^2 d\eta^2 + dR^2 + d\vec{x}_\perp^{\,2} \qquad (R>0)$$

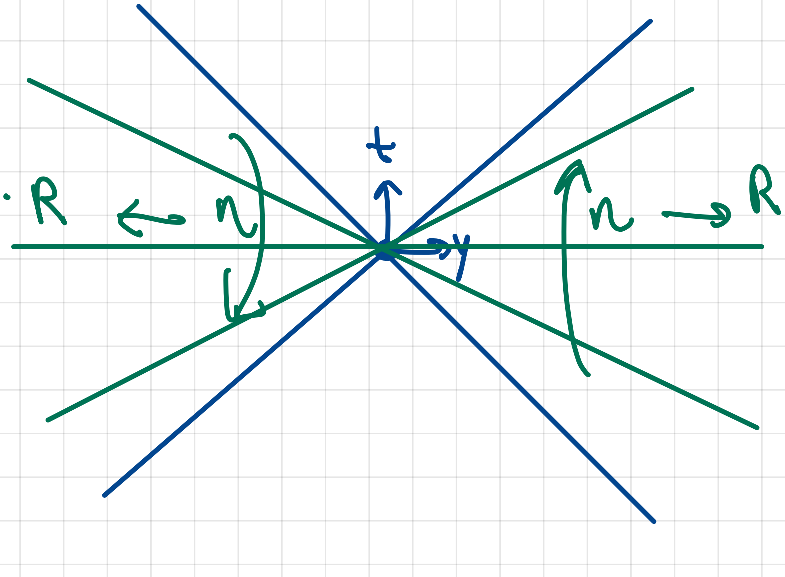
"Rindler Coordinates"

Related to Minkowski coords by

$$t = R \sinh \eta$$

$$y = R \cosh \eta$$

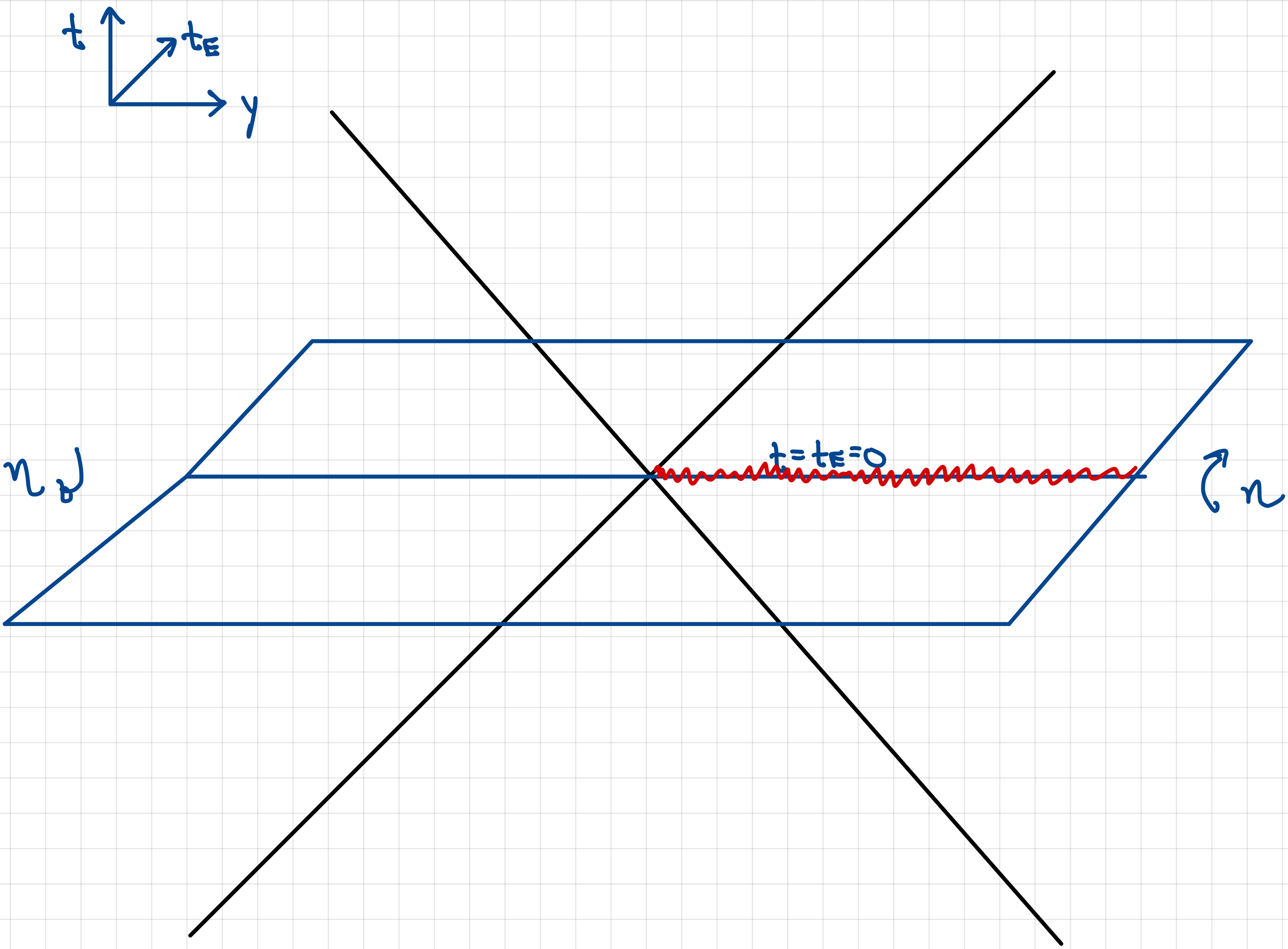
$$\to -dt^2 + dy^2 + d\vec{x}_\perp^{\,2}$$



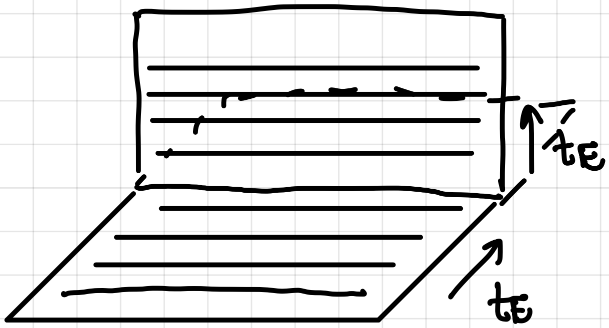
$R > 0 \Rightarrow$ coords cover <u>either</u> left or right wedge.

In Rindler coords, η is "time", K_A is "energy", and ρ_A is "thermal".

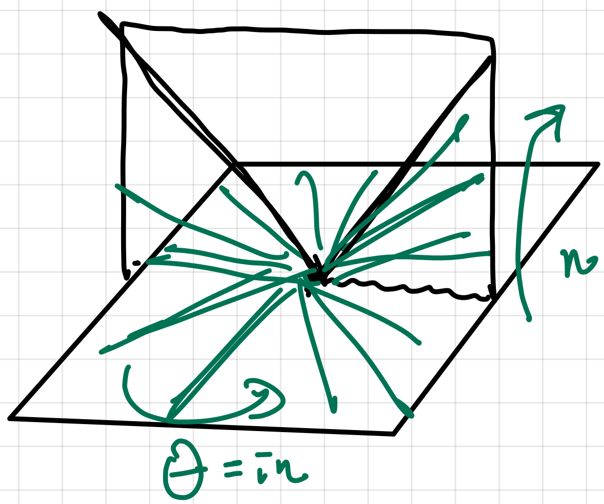
Euclidean/Lorentzian recap



If we slice this cartesian-ly, we get



If we slice it with polar/Rindler coords,



Physical consequences

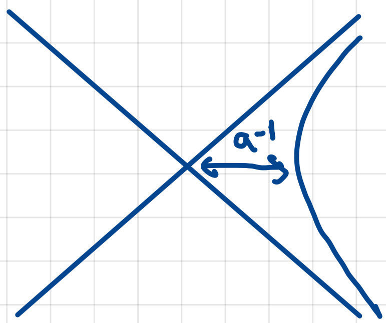
Uniformly accelerating observers really see a temperature

$$T_{obs} = \frac{a}{2\pi}$$ ← acceleration

worldline:

$$R = a^{-1}$$

$$\eta = a\tau \quad \text{(proper time)}$$



observer defines "energy" conjugate to $\partial_\tau = a^{-1}\partial_\eta$

so $E_{obs} = a K_A$

$$e^{-2\pi K_A} = e^{-\frac{2\pi}{a}E_{obs}} \quad \checkmark$$

thermometer!

numbers:

$$T_{obs} = \frac{\hbar a}{2\pi c k_B} \Rightarrow 1K \ @ \ 10^{20}\, m/s^2$$

This is a complete derivation. I (purposely) did not use the standard language of perturbative QFT, and the result applies even in strongly coupled QFT. Of course we can use $\hat{a}, \hat{a}^\dagger$ and this is done in the reading - I will only sketch it quickly in lecture.

Free fields (2D) (sketch!)

$$\Box \phi = 0$$

this has plane-wave solutions, similar to flat space

Rindler modes:

$$\phi_k = e^{-i\omega_k \eta + ik \log R}, \qquad \omega_k = |k|$$

$$\hat{\phi} = \sum_k \left(b_k \phi_k + b_k^\dagger \phi_k^* \right)$$

"Rindler vacuum"

$$b_k |0\rangle_R = 0 \qquad \forall k$$

particles -

$$b_{k_1}^\dagger b_{k_2}^\dagger \cdots |0\rangle_R \sim |n_{k_1}, n_{k_2}, \ldots\rangle$$

In Minkowski, we'd have a different mode expansion, and therefore different vacuum:

$$|0\rangle_R \neq |0\rangle_{\text{Mink.}}$$

Choice of "time"

→ defn. of "energy" and "particle"

→ defn. of "vacuum state"

mode expansions give same result

$$\rho_A = e^{-2\pi K_A}$$

$= \omega_k$ for Rindler modes, because

$$\partial_\eta \phi_k = -i\omega_k \phi_k$$

Comments

* $-\log \rho_A = 2\pi K_A$ is called the "modular Hamiltonian" (or "entanglement Hamiltonian")

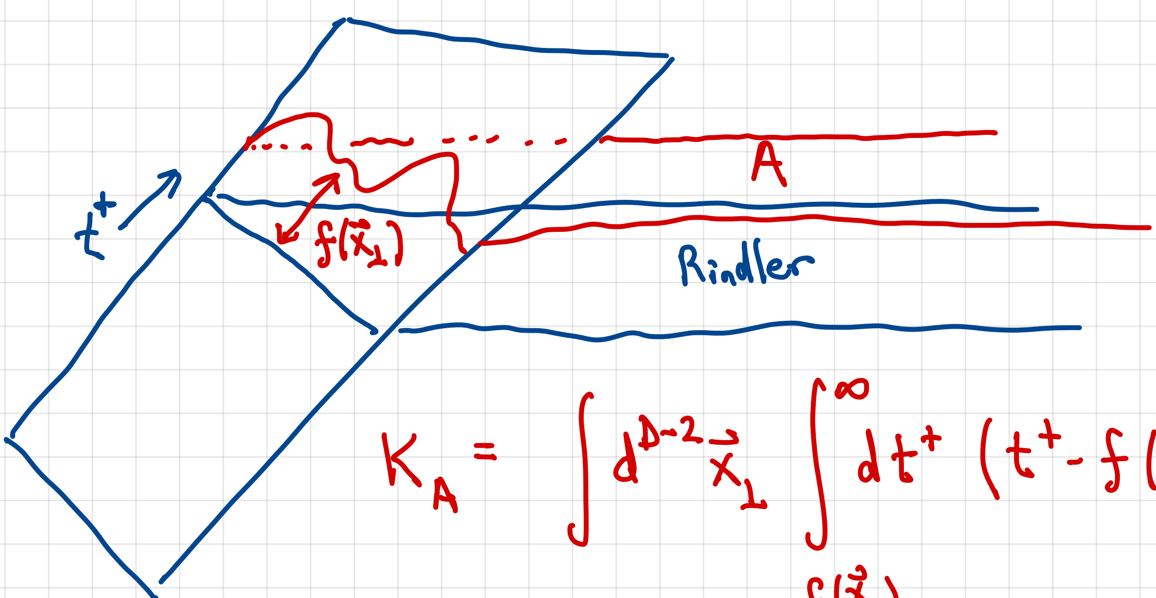
* Only in very special cases does

$$K_A = \int_A \text{"local}(x)\text{"}$$

- Rindler in vacuum
- Ball in vacuum in CFT
- 2d CFT interval in thermal state on $\mathbb{R}$

  $\underset{A}{\vdash\!\!-\!\!-\!\!\dashv}$

- "wiggly Rindler" in vacuum



$$K_A = \int d^{D-2}\vec{x}_\perp \int_{f(\vec{x}_\perp)}^{\infty} dt^+ \left(t^+ - f(\vec{x}_\perp)\right) T_{t^+t^+}$$

So what is K_A more generally? Something terrible and nonlocal. Known for 2 intervals in 2D fermion.

Periodicity Trick

Given a metric

$$ds^2 = -R^2 d\eta^2 + dR^2$$

we can quickly find temperature:

1) $\eta = -i\eta_E$

$$ds^2 = dR^2 + R^2 d\eta_E^2$$

2) Smoothness @ $R=0$

$\Rightarrow \eta_E \sim \eta_E + 2\pi$ ident.

(why? on smooth mf., circle with proper radius R has circumf. $2\pi R$ as $R \to 0$)

3) $\eta \sim \eta + 2\pi i$

recall time $\sim$ time $+ i\beta$

$\therefore$ $\boxed{\beta = 2\pi}$ (w.r.t. $Q[\partial_\eta] = K_A$)

This is telling you the temperature in the state smooth across the horizon.